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First passage times in homogeneous nucleation: dependence on the total number of particles

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Motivated by nucleation and molecular aggregation in physical, chemical and biological settings, we present an extension to a thorough analysis of the stochastic self-assembly of a fixed number of identical particles in a finite volume. We study the statistic of times it requires for maximal clusters to be completed, starting from a pure-monomeric particle configuration. For finite volume, we extend previous analytical approaches to the case of *arbitrary size-dependent* aggregation and fragmentation kinetic rates. For larger volume, we develop a scaling framework to study the behavior of the first assembly time as a function of the total quantity of particles.

We find that the mean time to first completion of a maximum-sized cluster may have surprisingly a very weak dependency on the total number of particles. We highlight how the higher statistic (variance, distribution) of the first passage time may still help to infer key parameters (such as the size of the maximum cluster) from data. And last but not least, we present a framework to quantify the formation of cluster of macroscopic size, whose formation is (asymptotically) very unlikely and occurs as a large deviation phenomenon from the mean-field limit. We argue that this framework is suitable to describe phase transition phenomena, as *inherent infrequent stochastic processes*, in contrast to classical nucleation theory.

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I. INTRODUCTION

The self-assembly of macromolecules and particles into cluster is a fundamental process in many physical, chemical and biological systems. Although particle nucleation and assembly have been studied for many decades^{1,2}, interest in this field has recently intensified due to engineering, biotechnological and imaging advances at the nanoscale level³⁻⁵. Applications range from material physics to cell physiology and virology (for a detailed list of examples see⁶ and references therein). Many of these applications involve a fixed “maximum” cluster size – of tens to hundreds of units – at which the process is completed or beyond which the dynamics change^{7,8}. One example include the rare self-assembly of mis-folded proteins into fibril aggregate in neurodegenerative diseases (Alzheimer, Parkinson, Prion...)^{9,10}. Developing a stochastic self-assembly model focusing on the formation of a fixed “maximum” cluster size is thus important for our understanding of a large class of biological processes, and the quantification of the variability of the experimental data¹¹⁻¹⁵.

Theoretical models for self-assembly have typically described mean-field concentrations of clusters of all possible sizes using the well-studied mass-action, Becker-Döring equations¹⁶⁻¹⁹. While Master equations for the fully stochastic nucleation and growth problem have been derived, and initial analyses and simulations performed²⁰⁻²⁴, there has been relatively less work on the stochastic self-assembly problem. It has been re-

cently shown that in finite systems, where the maximum cluster size is capped, results from mass-action equations are inaccurate and that in this case a discrete stochastic treatment is necessary^{6,25}. We consider here the Becker-Döring model (BD) defined by the following biochemical reactions

$$C_1 + C_i \xrightleftharpoons[q_{i+1}]{p_i} C_{i+1}, \quad i \geq 1, \quad (1)$$

where C_i denotes the number (or concentration) of clusters of size i . Note that in the stochastic version (SBD), the state-space of the system is discrete and finite (see Fig. 1), given by all possible combinations that have a given fixed total number of particles (defined by M , given by the initial condition)

$$\mathcal{E} := \left\{ (C_i)_{i \geq 1} \subset \mathbb{N} : \sum_{i \geq 1} i C_i = M \right\}. \quad (2)$$

The configuration $(C_i(t))_{i \geq 1}$ evolve in continuous time by discrete jumps according to Markovian description of the reactions (1). In our previous examination of first assembly time in this model⁶, we found that a striking finite-size effect arises in the limit of slow self-assembly. In particular, a *faster* detachment rate can lead to a *shorter* assembly time. This unexpected effect arise as the finite-size system may occupy some configurations that have been named “traps”, where no free particle is available and the maximal-size cluster completion may occur only through first a detachment of a particle from a cluster. Discrepancies between the mean-field mass-action approach and the stochastic model were indeed

where $\delta_k^1 = 1$ if $k = 1$ and $\delta_k^1 = 0$ if $k > 1$.

The first assembly time (FAT) for the stochastic discrete Becker-Döring is defined as a first passage time problem²⁹

$$T_{1,0}^{N,M} := \inf\{t \geq 0 : C_N(t) = 1\}. \quad (7)$$

Hence the FAT is the first time to obtain a cluster of size N , starting with a pure single particle initial state, with M particle (see Fig. 1 for an example). To link with macroscopic definition of the nucleation time, we will also consider the generalized first assembly time (GFAT) problems

$$T_{\rho,h}^{N,M} := \inf\{t \geq 0 : C_N(t) \geq \rho M^h\}, \quad (8)$$

for given positive constant ρ and $0 \leq h \leq 1$. Here, we want to analyze the behavior of $T_{\rho,h}^{N,M}$ when $M \rightarrow \infty$, for fixed N , and when both $M, N \rightarrow \infty$. This behavior will depend on scaling on the physical rates p, q , which may naturally depend on the total mass (or volume) of the system³⁰. One way of computing the distribution of first assembly times is to consider the Backward Kolmogorov equation (BKE) describing the evolution of the configuration probabilities as a function of local changes from the initial configuration. The BKE Approach was taken in⁶. It has the advantage to yields exact results for the full distribution of FAT, but it is strictly limited by the size of the system of equations, that grows exponentially with M . In this paper, we rely on exact calculation of simplified reduced models, limit theorems from Eq. (5) for large M and N , and extensive numerical simulations of these equations.

III. RESULTS AND ANALYSIS

Although the state-space (2) of the SBD model (1) is finite, the first passage problem defined by Eq. (8) is in general a very difficult problem: see preliminary studies in^{6,20,21,23}. There are two distinct simplifications that allow the problem to be analytically tractable. We present them here briefly in sections **A** and **B** (and generalize results from⁶). Then we present two asymptotic results for large volume, $M \rightarrow \infty$, with either finite or infinite nucleus size in sections **C** and **D**, respectively. The strategy will be based on re-scaling procedure of the stochastic Eq. (5). Numerical illustrations and results are postponed to the next section.

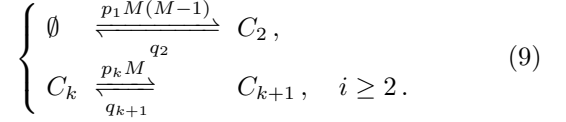
A. Constant Monomer formulation

The SBD model defined by Eq. (5) has the constant mass property

$$\sum_{k \geq 1} k C_k(t) \equiv \sum_{k \geq 1} k C_k(0) = M, \quad t \geq 0.$$

In the original formulation of the BD model (sometimes used in the deterministic context^{17,27}), the total mass

of the system is not preserved, but rather the quantity of free particles (think *e.g.* of a source/sink that will keep instantaneously constant the quantity of available free particles). We will refer to this formulation as the constant monomer stochastic Becker-Döring model (CMSBD). We can represent it by the following reactions



In the above formulation, $C_1(t) \equiv M$ is now a constant over time. Note that we expect such model to be close to the original SBD (for small times, up to the FAT) in the limit of large number of particle M . The main advantage of the constant monomer formulation is to be linear and hence analytically solvable. Indeed, it is known that for linear population model³¹, the number of individuals in each subclass of the population (starting with no individuals at time 0), namely here $C_2(t), \dots, C_N(t), \dots$, are independent Poisson random variable. Moreover, for this model (9), the mean $c_2(t), \dots, c_N(t), \dots$ are solution of the linear equation, given for all $t \geq 0$, by

$$\frac{d}{dt} c_k(t) = j_{k-1}(t) - j_k(t), \quad \forall k \geq 2, \quad (10)$$

with

$$\begin{cases} j_1(t) = p_1 M(M-1) - q_2 c_2(t), \\ j_k(t) = p_k M c_k(t) - q_{k+1} c_{k+1}(t) \quad \forall k \geq 2, \end{cases} \quad (11)$$

and initial condition $c_k(0) = 0$ for all $k \geq 2$. Note that the last set of Eq. (10)-(11) is very close to the deterministic Becker-Döring model (3)-(4) taking $c_1 \equiv M$. To calculate the FAT $T_{1,0}^{N,M}$ we use the survival function

$$\begin{aligned} S_{1,0}^{N,M}(t) &:= \mathbb{P}\{T_{1,0}^{N,M} > t\} \\ &= \mathbb{P}\{C_N(s) = 0, s \leq t \mid C_k(0) = M\delta_{k=1}\}. \end{aligned}$$

Then, using an absorbing boundary condition at $k = N$ ($q_N = p_N = 0$) together with the initial condition entail that $C_N(t) = 0$ for some $t \geq 0$ if and only if $C_N(s) = 0$ for all $s \leq t$, so that

$$S_{1,0}^{N,M}(t) = \mathbb{P}\{C_N(t) = 0 \mid C_k(0) = M\delta_{k=1}\}.$$

Finally, since $C_N(t)$ is Poisson distributed (linear system) with mean $c_N(t)$, we have

$$S_{1,0}^{N,M}(t) = e^{-c_N(t)}. \quad (12)$$

Equations (10)-(11) with the absorbing boundary at $k = N$ can be rewritten as a linear system

$$\begin{cases} \dot{\mathbf{c}} = \mathbf{A}\mathbf{c} + \mathbf{B}, \\ \dot{c}_N(t) = p_{N-1} M c_{N-1}(t), \end{cases} \quad (13)$$

where $\mathbf{c} = (c_2, c_3, \dots, c_{n-1})^T$, $\mathbf{B} = (p_1 M(M-1), 0, \dots, 0)^T$ and \mathbf{A} is a tridiagonal matrix with elements

$$\begin{cases} a_{k,k} = -q_{k+1} - p_{k+1}M, \\ a_{k+1,k} = p_{k+1}M, \\ a_{k,k+1} = q_{k+1}. \end{cases}$$

The study of the linear system (13) has been performed both for the infinite dimensional case³² and for the truncated case³³. In particular, it is shown that a similarity transformation

$$\mathbf{A} = \mathbf{DSD}^{-1}$$

with $D = \text{diag}(\sqrt{\tilde{Q}_k M^k})$ and $\tilde{Q}_k = \prod_{j=2}^k \frac{p_{j-1}}{q_j}$ leads to a matrix \mathbf{S} real symmetric tri-diagonal with non-zero elements on the sub and super-diagonal. Hence, classical linear algebra results shows that the eigenvalues of \mathbf{A} are real and distinct. Then, a general form of $c_N(t)$ is given by

$$c_N(t) = p_{N-1}M \left[\sum_{k=1}^{N-2} \alpha_k V_{N-2}^{(k)} \frac{e^{\lambda_k t} - 1}{\lambda_k} - (\mathbf{A}^{-1}\mathbf{B})_{N-2} t \right].$$

where λ_k, V^k denotes the eigenelements of \mathbf{A} and α_k are determined by initial conditions. Analytical solutions are available¹² for constant coefficient only (the matrix \mathbf{A} is in such case a Toeplitz matrix, with constant diagonal values, see Annex A.1). However, asymptotic expression are valid in the general cases. In particular, we have for small times $c_2(t) = p_1 M(M-1)t + o(t)$ and

$$\dot{c}_k = p_{k-1}M c_{k-1} + o(t),$$

so that, for $t \ll 1$,

$$c_N(t) \approx_{t \ll 1} M^N \prod_{k=1}^{N-1} p_k \frac{t^{N-1}}{(N-1)!},$$

and Eq. (12) is thus the survival function of a Weibull distribution, of shape parameter $k = N-1$ and scale parameter $\lambda = ((N-1)!/(M^N \prod_{k=1}^{N-1} p_k))^{1/(N-1)}$. Hence, we get

$$\langle T_{1,0}^{N,M} \rangle \approx_{M \rightarrow \infty} \frac{\Gamma(1 + 1/(N-1))}{\left(\prod_{k=1}^{N-1} p_k \right)^{1/(N-1)}} \frac{((N-1)!)^{1/(N-1)}}{M^{1+1/(N-1)}}. \quad (14)$$

Variance formula for the Weibull distribution yields the asymptotic coefficient of variation (standard deviation over the mean)

$$cv_{T_{1,0}^{N,M}} \approx_{M \rightarrow \infty} \sqrt{2(N-1) \frac{\Gamma(2/(N-1))}{\Gamma(1/(N-1))^2} - 1}. \quad (15)$$

Note in particular that the coefficient of variation do not vanish in large population, and that it is independent of

the particular shape of the aggregation rate and depends only on the size of the maximal cluster N .

For the generalized first assembly time GFAT, similar time scale asymptotic on the Eq. (13) on the mean gives the following expression

$$\langle T_{\rho,h}^{N,M} \rangle \approx_{M \rightarrow \infty} \frac{C(p,N)}{M} \frac{1}{M^{(1-h)/(N-1)}}, \quad (16)$$

where $C(p,N)$ is a constant that depends only on N and the aggregation rates $p_k, k \leq N$ (that can be made explicit if the full solution of Eq. (13) is known). Those asymptotic expressions are illustrated in Annex (Fig. A.1) where a perfect match is observed with numerical simulations.

B. Single cluster model

Another simplified model that can be analytically solved for the FAT problem is given by the assumption that a single cluster can be formed at a time^{6,34}. We expect such model to be close to the original model when the fragmentation dominates, so that formation of many (large) cluster is unlikely. In such model, called the single-cluster stochastic Becker-Döring (SCSBD) model we may represent only the size of the single cluster, so that the state space is now one dimensional, being simply

$$\mathcal{E}_1 := [1, \dots, N],$$

and the possible reactions are given by (k denotes the size of the single cluster)

$$\begin{cases} k = 1 \xrightleftharpoons[p_{k+1}]{p_1 M(M-1)} k = 2, \\ k \xrightleftharpoons[q_{k+1}]{p_k(M-k)} k+1, \quad k \geq 2. \end{cases} \quad (17)$$

In such a scenario, exact solution and classical First Passage Theory³⁰ gives (it is a one-dimensional discrete random walk)

$$\langle T_{1,0}^{N,M} \rangle = \sum_{i=1}^{N-1} \sum_{j=1}^i \frac{\prod_{k=j+1}^i q_k}{\prod_{k=j}^i p_k} \frac{1}{M^{\delta_j^1} \prod_{k=j}^i (M-k)}. \quad (18)$$

In addition, general formula for variance and cumulative distribution function are available³⁵. Those theoretical expressions are illustrated in Annex (Fig. A.2) where a perfect match is observed with numerical simulations.

Asymptotic expressions of the mean assembly time is straightforwardly deduced from Eq. (18). For instance, assume $q_k = \frac{\bar{q}_k}{\varepsilon}$ and that $\varepsilon \rightarrow 0$, the leading order of the mean assembly time is

$$\langle T_{1,0}^{N,M} \rangle \approx_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{N-2}} \frac{\prod_{k=2}^{N-1} \bar{q}_k}{\prod_{k=1}^{N-1} p_k \prod_{k=0}^{N-1} (M-k)}.$$

Also, one can show that in the asymptotic $\varepsilon \rightarrow 0$, for large fragmentation rate, the FAT $T_{1,0}^{N,M}$ is an exponential distribution⁶.

Finally, for large N and M , we can rescale the sum in Eq. (18) to obtain a suitable expression for the mean FAT when $N \rightarrow \infty$. Assume that the aggregation rates scale with M so that $p_1 = \bar{p}_1/M^2$, $p_k = \bar{p}_k/M$, $k \geq 2$. Then, let us introduce the rescaled size variable $x = k/N$, and define the rescaled kinetic rate

$$\begin{aligned}\bar{p}(x) &= \sum_{k \geq 2} \bar{p}_k \mathbf{1}_{[k/N, (k+1)/N)}(x), \\ q(x) &= \sum_{k \geq 2} q_k \mathbf{1}_{[k/N, (k+1)/N)}(x),\end{aligned}$$

we have, for $N = \sqrt{M} \rightarrow \infty$ (see details in Annex, section A.2.2),

$$\langle T_{1,0}^{\sqrt{M},M} \rangle \approx_{M \rightarrow \infty} M \int_0^1 \int_0^y \frac{e^{(y^2-z^2)/2}}{q(y)} \cdot \exp \left[\sqrt{M} \int_z^y \ln \left(\frac{q(x)}{\bar{p}(x)} \right) dx \right] dy dz. \quad (19)$$

In particular, when $q(x) > \bar{p}(x)$ on an interval of positive measure on $[0, 1]$, the last expression (19) implies that the mean FAT to reach the *macroscopic* size $x = 1$ ($k = N = \sqrt{M}$) is exponentially large as $M \rightarrow \infty$. As an example, suppose that $q > \bar{p}$ are size-independent. Then, Eq. (19) becomes

$$\langle T_{1,0}^{\sqrt{M},M} \rangle \approx_{M \rightarrow \infty} \frac{M}{q} \int_0^1 \int_0^y e^{(y^2-z^2)/2} \left(\frac{q}{\bar{p}} \right)^{\sqrt{M}(y-z)} dy dz.$$

Those theoretical expressions are illustrated in Annex (Fig. A.3 and A.4). Note that a different approach is to link the one-dimensional discrete random walk (17) with a one-dimensional stochastic differential equation, and to use Large Deviation Theory to derive asymptotic FAT³⁴ (Annex, section A.2.3). This scaling approach and the link with a continuous size model when $N \rightarrow \infty$ will be taken on the full SBD model in section **D**.

C. Large M , finite N

In this section, we investigate the behavior of the SBD and its FAT when the total number of particles M tends to infinity, while the size N of the maximal cluster to reach stay finite. We distinguish two scenarios, which yields distinct results. In the first one, the aggregation and fragmentation rate p_k, q_k are taken independent of M . As the aggregation propensities increase with M , it is expected that the FAT decrease to 0 as $M \rightarrow \infty$. The objective is to find valid asymptotic expressions, and its dependence with respect to other parameters, like the maximal cluster size N for instance. In the second case, the aggregation rate is scaled with the total number of particles, with $p_k = \frac{\overline{p}_k}{M}$. This scaling is motivated by classical

system size expansion of chemical reaction networks³⁰. As the total number of particles increases, the volume also increases so that the overall reaction propensities of the aggregation reactions stay constant. In such case, one expect to recover the deterministic first passage time of the classical deterministic BD model.

Let us now introduce our general rescaling strategy. The number of cluster of size k , given by C_k , are rescaled into

$$D_k^M(t) = \frac{C_k(t/M^\gamma)}{M}$$

with γ a scaling coefficient to be chosen latter. Then, from Eq. (5)-(6), we obtain, for any $t \geq 0$,

$$\begin{cases} D_1^M(t) = 1 - 2J_1^M(t) - \sum_{k \geq 2} J_k^M(t), \\ D_k^M(t) = J_{k-1}^M(t) - J_k^M(t), \quad k \geq 2, \end{cases} \quad (20)$$

with

$$J_k^M(t) = \frac{1}{M} Y_k^+ \left(\int_0^t M^{2-\gamma} p_k D_1^M(s) (D_k^M(s) - M^{-1} \delta_k^1) ds \right) \\ - \frac{1}{M} Y_{k+1}^- \left(\int_0^t M^{1-\gamma} q_{k+1} D_{k+1}^M(s) ds \right), \quad k \geq 2. \quad (21)$$

We recall a standard result of convergence of Poisson Processes (law of large numbers³⁶), that

$$\frac{1}{n}Y(nt) \xrightarrow{n \rightarrow \infty} t,$$

where Y is a standard Poisson Process.

1. *No scaling of the aggregation rate*

Using $\gamma = 1$, and the standard law of large numbers applied to the Eq. (20)-(21), we can show³⁷ (see Annex, section 3) that the sequence of stochastic processes $(D_k^M(t))$ converges, as $M \rightarrow \infty$, in a rigorous sense (in the trajectory space) to the solution of the irreversible aggregation deterministic model (BD with $q_k = 0$), given, for all $t \geq 0$, by

$$\begin{cases} \frac{d}{dt}d_1 = -2j_1(t) - \sum_{k \geq 2} j_k(t), \\ \frac{d}{dt}d_k = j_{k-1}(t) - j_k(t), \quad \forall k \geq 2, \end{cases} \quad (22)$$

with

$$j_k(t) = p_k d_1 d_k(t), \forall k \geq 1, \quad (23)$$

and initial condition $d_1(0) = 1$ and $d_k(0) = 0$, for all $k \geq 2$. Intuitively, in the rescaled variable D_k^M , the aggregation process is much more favorable compared to the fragmentation because the number of free particles is very large. By definition of the GFAT Eq. (8), with $h = 1$,

$$MT_{\rho,1}^{N,M} = \inf\{t \geq 0 : D_N^M(t) \geq \rho\}.$$

Then, using the convergence of $(D_k^M(t))$, we obtain the following asymptotic behavior of the GFAT for $h = 1$,

$$\lim_{M \rightarrow \infty} MT_{\rho,1}^{N,M} = \inf\{t \geq 0 : d_N(t) \geq \rho\}. \quad (24)$$

The latter quantity is deterministic, and may be finite or infinite, according to the respective value of p_k , N and ρ .

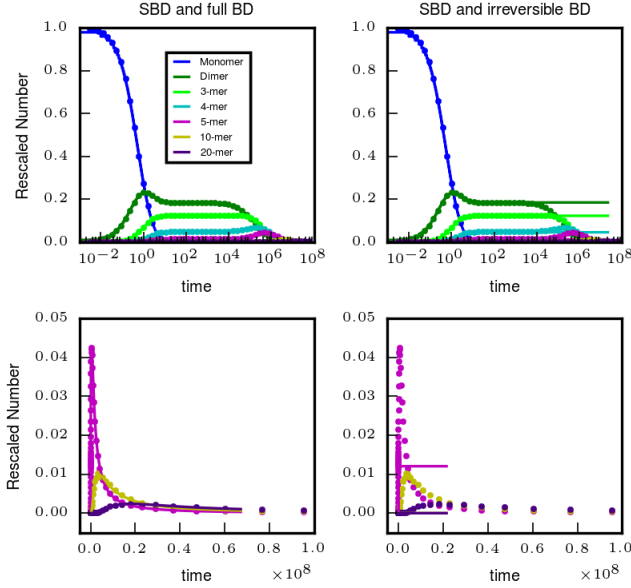


FIG. 2: SBD and BD Trajectories. For the SBD, we simulate the rescaled Eq. (20)-(21), with $M = 10^{5.5}$, and kinetic rates are $p_k \equiv 1$ for all $k \geq 1$ and $q_k \equiv 1$ for all $k \geq 2$. For the BD model, we simulate on the left columns the full BD Eq. (22)-(26) and on the right columns the irreversible BD Eq. (22)-(23). The rescaled SBD trajectories are plotted with filled circles, together with the corresponding BD trajectories in plain lines, for the monomer and i -cluster, $i = 2, 3, 4, 5, 10, 20$, according to the legend. The lower panel correspond to the same numerical simulation of the upper panel, with a zoom on the y -axis to improve the visualization of the i -cluster for $i = 5, 10, 20$. It is immediate to see that the full BD Eq. (22)-(26) agrees perfectly with the rescaled SBD Eq. (20)-(21) for all time, while the irreversible BD Eq. (22)-(23) matches only up to a time scale of order M .

The limit model (22)-(23) do not capture the FAT and the GFAT $T_{\rho,h}^{N,M}$ for $h < 1$ (such event is reached for time $t = 0^+$). However, as the initial number of monomers is large, we can use an intermediate approximation of the dynamic of the stochastic model, as a hybrid deterministic/stochastic model. First, note that pure coagulation BD model (22)-(23) have been extensively studied³⁸, where exact time-dependent solution for $p_k = pk$ are given, and time asymptotic behavior are given for power law coefficient $p_k = pk^\lambda$, $0 \leq \lambda \leq 1$. We restrict the following discussion to the constant rate case, $\lambda = 0$, for simplicity (results are analogous in the power law case). In such case, the stationary state of the pure coagulation

BD model^{17,38} (22)-(23) is $d_1^* = 0$ and

$$d_k^* = \frac{k-1}{ek!}, \quad k \geq 2. \quad (25)$$

Although the rescaled threshold ρM^{h-1} will be reached by d_N (and hence by D_N^M) for any ρ and $h < 1$ for large enough M (as $d_N^* > 0$), one can already see that for “intermediate” M , we may have $Md_N^* \ll 1$, so that the threshold may not have been reached while the free particle have vanished ($d_1^* = 0$). In such case, it is necessary to take into account the small but crucial contribution of the aggregate shortening. For that, let us consider as a further approximation of Eq. (20)-(21) the following deterministic model (still with constant rate coefficients, to simplify the following), given, for all $t \geq 0$, by Eq. (22) and flux definition

$$j_k(t) = pd_1(t)d_k(t) - \frac{1}{M}qd_{k+1}(t), \quad k \geq 1, \quad (26)$$

where $1/M$ is seen as a small parameter. To obtain results for the quantity $T_{\rho,h}^{N,M}$ for any $h < 1$, we need to study the time-dependent properties of the favorable aggregation limit $M \rightarrow \infty$ of the deterministic BD model Eq. (22)-(26). The following discussion is illustrated with Fig. 2 (see also in Annex, Fig. A.7, A.8). For constant rate p, q , it is known¹⁷ that under favorable aggregation limit $q/M \ll p$, the deterministic BD model Eq. (22)-(26) exhibits the following successive periods:

- Firstly, the model behaves as the irreversible aggregation BD model, Eq. (22)-(23), during a time-scale of order $e \log(M)$, until monomer concentration $d_1(t)$ becomes small;
- Secondly, when the monomer concentration d_1 is of order $1/M$, there is a metastable period in which each concentration species of size $k \geq 2$ are nearly constant, equal to d_k^* , the equilibrium state Eq. (25) of the irreversible aggregation BD model, Eq. (22)-(23). The concentration $d_k(t)$ stays roughly constant to the values d_k^* , distinct from the steady-state values of the full BD Eq. (22)-(26), until the next time scale;
- Thirdly, at a time scale of order M (which is the time scale of aggregate shortening), larger aggregates are created within a process akin to diffusion in the size k -space (slow redistribution of aggregate sizes);
- Finally, every concentration species d_k reaches the classical steady-state value of the full BD Eq. (22)-(26) within a time scale of order M^2 . Steady-state values \bar{d}_k are given by

$$\bar{d}_k = \left(\frac{pM}{q}\right)^{k-1} \bar{d}_1^k, \quad k \geq 2,$$

where \bar{d}_1 is determined by the mass conservation property. Such values can be approximated by $\bar{d}_1 \approx 1/M$ and $\bar{d}_k \approx 1/M(1 - 1/\sqrt{M})^{k-1}$.

To approximate the GFAT $T_{\rho,h}^{N,M}$, we need to know in which of these periods the event $\{C_N(t) \geq \rho M^h\}$ is

reached. This mostly depends on the critical size N of the nucleus as follows. If M is large enough, then the metastable state is large enough, *i.e.* $c_N^* = Md_N^* > \rho M^h$, and the cluster number $C_N(t)$ will reach the threshold during the pure-aggregation time-scale ($\log(M)/M$ in the original time scale), and the GFAT $T_{\rho,h}^{N,M}$ is found (see numerical section) to behave as the linear CMSBD model (9) with $C_1 = M$.

In the opposite case, for intermediate M and large enough N such that $c_N^* \ll \rho M^h$, we expect $C_N(t)$ to reach the threshold after the metastable period (of order 1 in the original time scale). As the initial pure-aggregation phase is short ($\log(M)/M$), we can neglect it, and use the metastable values c_k^* as initial values for a linear CMSBD model (9) where the monomer number is now equal to $C_1 \equiv c_1^*$ (see Annex, section 3.1 and Fig. A.10) given by¹⁷

$$c_1^* = q \frac{c_2^* + \sum_{k=2}^{N-1} c_k^*}{p \sum_{k=2}^{N-1} c_k^*} = \frac{3}{2} \frac{q}{p}.$$

Hence c_1^* is independent of the initial number of monomers M and is of order q/p . Thus the GFAT depends on M only through the initial condition c_k^* , $k \geq 2$, and is found to be (see numerical results) almost independent of M on several order of magnitude for $N \geq 15$. Finally, note that there is always a (small) probability that the threshold is reached before the metastable period, which is responsible for a bimodal behavior of $T_{\rho,h}^{N,M}$ (see numerical results). For values of d_k^* and a summary of the different cases, see Tables II and I.

Finally, performing a second-order approximation of Eq. (20)-(21), we obtain a system of stochastic differential equation with variance of order $\sqrt{1/M}$ (see details in Annex, section A.3.2), and the GFAT can be computed by

$$MT_{\rho,h}^{N,M} \approx \inf\{t \geq 0 : \tilde{D}_N^M(t) \geq \rho M^{h-1}\},$$

where (\tilde{D}_k^M) denotes the solution of the second order stochastic differential equations. Such approach unfortunately do not provide analytical hints on the behavior of the GFAT, but provide a convenient tool to compute numerically an approximation of the GFAT for very large M , where the exact MC simulations slow down (see numerical results).

2. "System-size expansion" scaling

The above asymptotic approximation have been performed assuming that the reaction rates are independent of M . However, the limit $M \rightarrow \infty$ may be understood as a system-size expansion, in which case reaction rates must be scaled with the system size according to their respective order. In particular, it is classical³⁰ that first-order reaction rates are independent of the system size, and second-order reaction rates are inversely proportional to the system size. Thus we are led to use $p_k = \frac{\bar{p}_k}{M}$.

With $\gamma = 0$, the re-scaled variable $D_k^M(t) = C_k(t)/M$ converges now to the BD system given, for all $t \geq 0$, by Eq. (22) and flux definition

$$j_k(t) = \bar{p}_k d_1 d_k(t) - q_k d_{k+1}(t), \quad k \geq 1. \quad (27)$$

As before, using the convergence of $(D_k^M(t))$, we obtain the following asymptotic behavior of the GFAT for $h = 1$,

$$\lim_{M \rightarrow \infty} T_{\rho,1}^{N,M} = \inf\{t \geq 0 : d_N(t) \geq \rho\}.$$

Once again, the latter quantity is deterministic, and may be finite or infinite, according to the respective value of q_k, \bar{p}_k, N and ρ . The GFAT $T_{\rho,h}^{N,M}$ with $h < 1$ behaves asymptotically as the GFAT of the linear CMSBD model (9) with $C_1 \equiv M$ and $p_k = \frac{\bar{p}_k}{M}$. Thus,

$$T_{\rho,h}^{N,M} \approx_{M \rightarrow \infty} C(\bar{p}, N) \frac{1}{M^{(1-h)/(N-1)}}, \quad (28)$$

where $C(\bar{p}, n)$ is a constant that depends only on N and $\bar{p}(k), k \leq N$. Second-order approximation may also be derived using a central limit theorem for D_k^M (see Annex, section A.3.2).

D. Large M and Large N

In this section, we investigate the behavior of the SBD and its FAT when the size N of the maximal cluster is large, and scales with the total number of particles M . As in section B, we will then naturally use the rescale size variable $x = k/N$. We distinguish again two scenario, which yields distinct results. In the first one, the aggregation and fragmentation rates p_k, q_k are taken independent of M . In the second one, the aggregation rates are scaled with the total number of particles, with $p_k = \frac{\bar{p}_k}{M}$. In both cases, a rescaling of the solution is found to be solution of a deterministic continuous size model, namely the Lifshitz-Slyozov model (LS). Indeed, we have detailed in a companion paper³⁹ how to choose a proper scaling and how to derive the limit equation for that rescaled solution. We show here the consistency of this scaling with the behavior of the GFAT.

We will restrict for simplicity here to the case $N = \sqrt{M}$. In that case, we define the rescaled measure on \mathbb{R}^+ ,

$$\mu^M(t, dx) = \sum_{k \geq 2} \frac{C_k(t/M^\gamma)}{\sqrt{M}} \delta_{k/\sqrt{M}}(dx), \quad (29)$$

and $C_1^M(t) = C_1(t/M^\gamma)/M$, where $\delta_x(\cdot)$ is the Dirac measure at x . The GFAT $T_{\rho,h}^{\sqrt{M},M}$ involves a larger and larger maximal size \sqrt{M} , which is rescaled to the macroscopic size $x = 1$ by the definition of the measure μ^M in Eq. (29). We also need to define accordingly macroscopic aggrega-

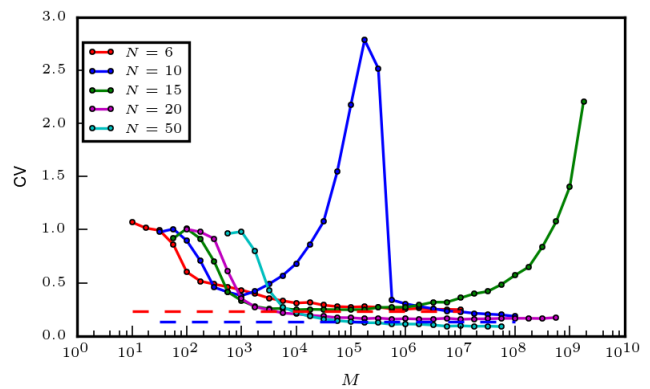


FIG. 3: First Assembly Time $T_{1,0}^{N,M}$ for the original SBD (section III C 1) as a function of the total mass M (in log-log scale) for five different maximal cluster sizes $N \in \{6, 10, 15, 20, 50\}$. Each color light dot is a single realization of the FAT. For each condition, large circles represent the statistical mean over 1000 samples (A few condition are sampled only once, namely for $N = 15, 20, 50$ and large M , for which the mean is not shown). Black dash-dotted lines are straight lines of slope -1 , color dash-dotted lines are straight lines of slope $-(1 + 1/(N - 1))$ (as in Eq. (14)). And for $N = 15, 20, 50$ we plot additionally dashed lines of slope resp. $-0.26, -0.15$ and -0.10 . The last panel in bottom-right represent the 5 mean FAT on the same scale. Kinetic rates are $p_1 = 0.5$, $p_k \equiv 1$, and $q_k \equiv 100$ for all $k \geq 2$.

We begin with the analysis of the FAT as a function of the total number of monomer M , when the maximal cluster size N and the aggregation rates p_k are fixed. For $N = 6, 10, 15$, the prediction of the asymptotic behavior of $T_{1,0}^{N,M}$, the waiting time for a single maximal cluster to be formed, is verified: the mean FAT decrease linearly in log-log scale as M increase, with a slope equal to $-(1 + 1/(N - 1))$, as in the linear CMSBD model (Fig. 3, upper panels), see Eq. (14). The coefficient of variation (cv, standard deviation over the mean), that measures the variability of the FAT, is also consistent with a transition from an exponential distribution to a Weibull distribution as M increases: the cv decreases from 1 to the predicted value by Eq. (15) (Fig. 4, and Fig. A.1 in Annex for the CMSBD). Furthermore, one can observe very clearly for $N = 15$ the bimodal behavior predicted for large but intermediate M values (Fig. 3, third panel). For M from 10^6 to 10^{10} , the sampled FAT

Similar results are obtained for the GFAT $T_{\rho,h}^{N,M}$, where the linear log-dependence with a slope $-(1+(1-h)/(N-1))$ (see Eq. (16) for the CMSBD model) is found to be perfectly satisfied for $N = 3, 5$ and $h = 0.25, 0.5, 0.75$ and

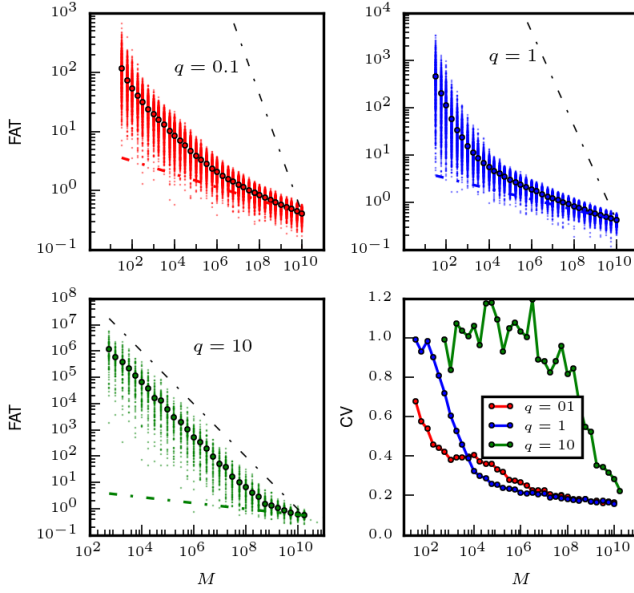


FIG. 5: First Assembly Time $T_{1,0}^{N,M}$ for the rescaled SBD (section III C 2) as a function of the total mass M (in log-log scale) for three different detachment rates $q \in \{0.1, 1, 10\}$, and $N = 10$. Kinetic rates are $p_1 = 0.5$, and $p_k \equiv 1$ and $q_k \equiv q$ for all $k \geq 2$. Each color light dot is a single realization of the FAT. For each condition, large circles represent the statistical mean over 1000 samples. Black dash-dotted lines are straight lines of slope -1 , color dash-dotted lines are straight lines of slope $-1/(N-1)$. Finally, the panel in bottom right represent the Coefficient of Variation (CV) as a function of the total mass M corresponding to the realizations of the first three panels (top and bottom left).

$h = 1$ (Annex, Fig. A.5, upper panels). Bimodal behavior and nearly flat log-dependence of the GFAT $T_{\rho,h}^{N,M}$ as a function of M on a broad range of M values is also observed for $N = 10, 20$ (Annex, Fig. A.5, lower panels). The size of the 'bimodal' region is found to be increased with increasing h (and N). For $N = 10, 20$ and $h = 1$, the mean FAT is increasing to ∞ as the deterministic limit given by Eq. (24) is infinite. The cv are non-monotonic with respect to M with a peak corresponding to the bimodal behavior (Annex, Fig. A.6). We show that the GFAT has a lower variability as h increase, and vanish for $h = 1$ and large M , in agreement with the deterministic limit in Eq. (24).

B. The First Assembly Time is non-monotonic with respect to the detachment rate

We verify in Annex, Fig. A.12, A.13 and A.14 the dependence of the FAT on the detachment rate (see also⁶). We confirm that the bimodal behavior is observed for *small* detachment rate, and that the mean FAT (and the cv) is a non-monotonic function of the detachment rate.

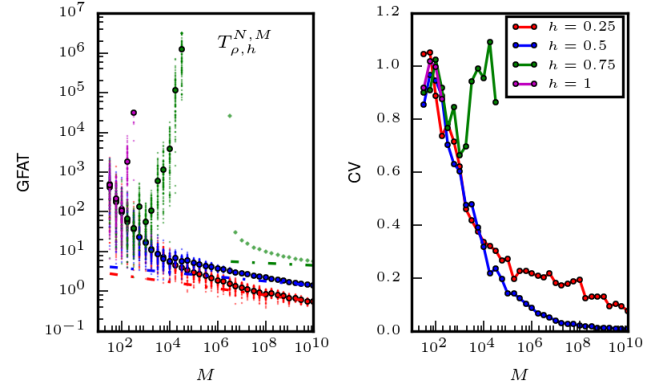


FIG. 6: (Left) Generalized First Assembly Time $T_{\rho,h}^{N,M}$ for the rescaled SBD (section III C 2) as a function of the total mass M (in log-log scale) for $h \in \{0.25, 0.5, 0.75, 1\}$, and $N = 10$. Kinetic rates are $p_1 = 0.5$, $p_k \equiv 1$ and $q_k \equiv 1$ for all $k \geq 2$. Each color light dot is a single realization of the GFAT. For each condition, large circles represent the statistical mean over 1000 samples (A few condition are sampled only once, namely for $h = 0.75$ and large M , for which the mean is not shown). Color dash-dotted lines are straight lines of slope $-(1-h)/(N-1)$. (Right) Coefficient of Variation (CV) as a function of the total mass M corresponding to the realizations of the left panel.

C. The Generalized First Assembly Time may increase with M for the system-size scaling

When the aggregation rates p_k are rescaled with the total number of monomer M (see section III C 2), the FAT to reach a maximal cluster of fixed size N decrease monotonically with M , and asymptotically with a linear log-dependence with a slope $1/(N-1)$ (Fig. 5), as predicted by Eq. (28). The same is valid for the GFAT $T_{\rho,h}^{N,M}$, for $h < 1$, with a slope $(1-h)/(N-1)$ (Fig. 6). However, for $h = 1$, if the threshold ρ is too large, the GFAT is never reached by the deterministic BD model (22)-(27). Thus, for the finite SBD, the GFAT for $h = 1$ increases to ∞ as M increases to ∞ . For $h = 0.75$, we also found that the GFAT is *non-monotonic* with respect to the total number of monomers, even though it converges to 0 for (very) large number of monomers.

D. Exponentially Large FAT for large maximal cluster size N and phase-transition phenomena

Finally, we verify in Fig. 7 and Fig. 8, that for large maximal size N , of order \sqrt{M} , the two scalings show in Eq. (32)-(34) are valid. Specifically, in Fig. 7, we see that for $M > 10^6$, the FAT is nearly deterministic and can then be predicted by the limit model Eq. (31). The same threshold is empirically observed in Fig. 8 for the GFAT as well. However, considering $p(x) = x$ and $q(x) = 1$, in Fig. 9, we show that exponential large devi-

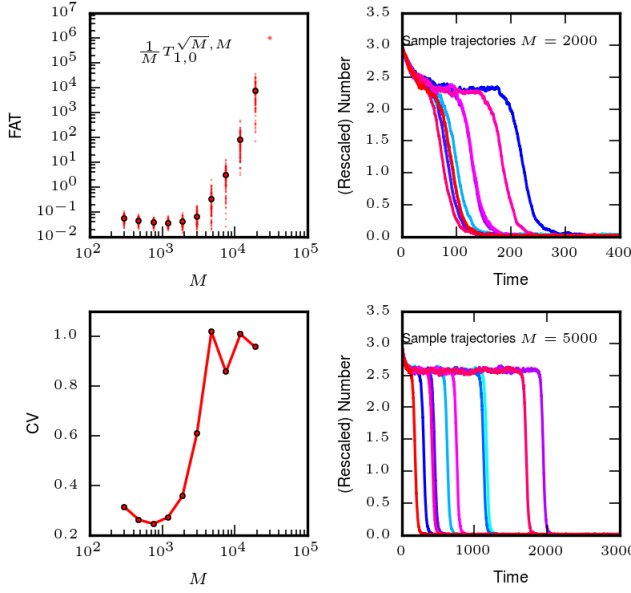


FIG. 9: (Top left) First Assembly Time $T_{1,0}^{\sqrt{M},M}$ for the rescaled SBD and large maximal cluster size of order $N = \sqrt{M}$ (section IIID 2) as a function of the total mass M (in log-log scale). Kinetics rates are $p(x) = x$ and $q(x) = 1$. (Bottom Left) Coefficient of Variation (CV) as a function of the total mass M corresponding to the realizations of the upper left panel. (Top and Bottom Right) Time-dependent trajectories of the rescaled number of monomers $c_1(t) = C_1(t)/M$, for $M = 2000$ (top) and $M = 5000$ (down). Each color line represent a single realization with the same initial condition and kinetic parameter.

namely the discrete-size Becker-Döring (BD) model and the continuous-size Lifshitz-Slyozov (LS) model. This has for first implication to be able to find very quickly the order of magnitude of the FAT (resp. GFAT) with the help of a single (fast) numerical simulation of a deterministic model (rather than by extensive numerical simulation of the full SBD model). With the help of a careful time scale analysis on the deterministic BD model, and with extensive numerical simulation, we also pointed out surprising deviations from the mean field deterministic model. First, for sufficiently large maximal cluster size ($N \geq 15$), the mean FAT is found to be very weakly-dependent on the total number of monomers M , and so for several order of magnitude of “intermediate values” of M (from 10^6 to 10^{13} in our simulations). The full distribution of the FAT is bimodal on this parameter region. We explained and gave practical criteria to observe this phenomena by a careful inspection of the metastable state of the favorable aggregation limit for the deterministic BD model. Second, for large maximal cluster size, we confirmed that for unfavorable aggregation kinetic ($q(x) > p(x)$), the mean FAT is exponentially large as M increases to ∞ for the full SBD model. We linked this behavior with phase-transition phenomena, where the number of monomers drastically drops to 0

in a very short time, compared to the FAT. This phase-transition phenomena occurs as a large deviation from the deterministic model, which predicts that the number of monomers stays constant (no aggregation takes place).

This study has generalized previous study on the first passage time on the stochastic Becker-Döring model. Up to our knowledge, this study is the first one to capture the behavior of the FAT and its generalization for arbitrary kinetic rates, and to explore systematically its dependence with respect to the total number of monomers and the size of the maximal cluster. Taking into account size-dependent kinetic rate is important in practice, as monomer binding and unbinding usually depends on the available surface area of the cluster (for the spherical shape, $p_k \sim k^{2/3}$). This study may have several important applications. One of this is the explanation of the nucleation time observed in *in vitro* polymerization assay of misfolded proteins linked to neurodegenerative diseases^{11–15}. Typical experiments performed in this field are able to record the nucleation time (defined as the time for which the polymerization starts) for various initial quantity of proteins. Some experiments have described a very weak dependence with respect to this initial quantity, where traditional nucleation theory could not explain this fact. Our stochastic approach points out several new behavior that may explain the observations. Furthermore, we argue that having a model that is able to take into account the observed variability on the nucleation time will be important for parameter inference from experimental data (see also the recent preprint⁴⁵). Indeed, even though the mean FAT may be weakly dependent on the maximal cluster size N (consider the slope of $1 + 1/(N - 1)$ for large M), having the observation of the full distribution will facilitate the inference of the maximal cluster size (the shape parameter of the Weibull distribution is $k = N - 1$). Finally, on a more theoretical side, the phase-transition phenomena of the SBD model for unfavorable aggregation and large cluster size seems to be described here for the first time. This gives a possible different definition of the nucleation rate, as an inherent infrequent stochastic process, in contrast to classical nucleation theory. It remains in the future to make a link with studies on gelation phenomena, which happen when a fraction of the mass is concentrated in a giant particle (N is of order of M). Such studies have been performed mostly in general smoluchowski coagulation models^{37,46,47}.

A number of generalization of this model could be considered and will be relevant to tackle new biophysical problems. One could generalize this study to allow general coagulation-fragmentation between any two clusters⁴⁸. This extension as well as the treatment of heterogeneous nucleation and secondary pathways will be considered in future work.

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